12.4. The cross product

Def (1) The determinant of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} := ad - bc$$

(2) The cross product (or vector product) of

$\vec{v} = (a_1, b_1, c_1)$ and $\vec{w} = (a_2, b_2, c_2)$ is

$$\vec{v} \times \vec{w} := (b_1 c_2 - b_2 c_1,\ -a_1 c_2 + a_2 c_1,\ a_1 b_2 - a_2 b_1)$$

$$= \left(\det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix},\ -\det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix},\ \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \right)$$

don't forget

Rmk (1) The output of a cross product is a vector.

(2) The cross product is defined only for 3-dimensional vectors.

Prop (Algebraic properties of the cross product)

(1) $\vec{v} \times \vec{v} = \vec{0}$.

(2) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

(3) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

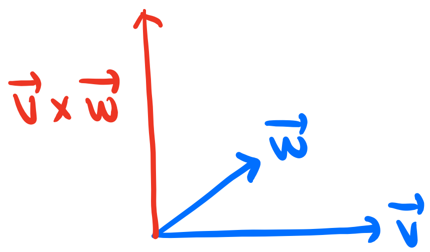
(4) $(r\vec{v}) \times \vec{w} = r(\vec{v} \times \vec{w})$ for any number r.

(5) $\vec{v} \times \vec{0} = \vec{0}$

Thm (1) $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$

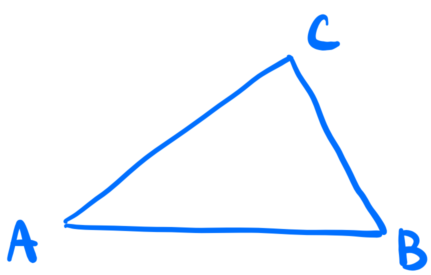
where θ is the angle between $\vec{v}$ and $\vec{w}$.

☆ (2) The direction of $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$, given by the right hand rule.



☆ **Cor** The area of the triangle ABC is

$$\frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}|\vec{BA} \times \vec{BC}| = \frac{1}{2}|\vec{CA} \times \vec{CB}|.$$



☆☆ **Thm** $\vec{v}$ and $\vec{w}$ are perpendicular if and only if $\vec{v} \cdot \vec{w} = 0$.

(Explanation: $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$

$\vec{v} \cdot \vec{w} = 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}$)

Ex Find two unit vectors which are orthogonal to $\vec{v} = (-2, 0, 2)$

$\underline{\text{orthogonal}} \;\; \| \;\; \text{perpendicular}$

Sol 1 (Cross product)

Idea : Choose a random vector $\vec{w}$ and take the unit vector of $\underbrace{\vec{v} \times \vec{w}}_{\text{perpendicular to } \vec{v}}$.

Take $\vec{w} = (1, 0, 0)$

$$\Rightarrow \vec{v} \times \vec{w} = (-2, 0, 2) \times (1, 0, 0) = (0, 2, 0)$$

$$|\vec{v} \times \vec{w}| = \sqrt{0^2 + 2^2 + 0^2} = 2.$$

The unit vectors are

$$\pm \frac{\vec{v} \times \vec{w}}{|\vec{v} \times \vec{w}|} = \pm \frac{1}{2}(0, 2, 0) = \boxed{\pm (0, 1, 0)}$$

Note This method works as long as you choose a vector $\vec{w}$ which is not parallel to $\vec{v}$.

If $\vec{w}$ is parallel to $\vec{v}$, then you get $\vec{v} \times \vec{w} = \vec{0}$ and thus find no unit vectors.

To see whether two vectors are parallel, you compare the ratios of their coordinates.

e.g. $(2, 1, 3)$ is parallel to $(4, 2, 6)$, but not to $(6, 3, 4)$

Sol 2 (Dot product)

$\vec{v}, \vec{w}$ perpendicular

Idea: Find a vector $\vec{w}$ with $\vec{v}\cdot\vec{w} = 0$,

then take the unit vector of $\vec{w}$.

Set $\vec{w} = (a, b, c)$.

$\Rightarrow \vec{v} \times \vec{w} = (2, 0, -2)\cdot(a, b, c) = 2a - 2c$.

We want $\vec{v}\cdot\vec{w} = 0$

$\rightsquigarrow 2a - 2c = 0 \rightsquigarrow a = c$.

Take $a = c = 0$, $b = 1$

* You can take any a, b, c with $a = c$.

$\Rightarrow \vec{w} = (0, 1, 0)$

$|\vec{w}| = \sqrt{0^2 + 1^2 + 0^2} = 1$.

The unit vectors are

$$\pm \frac{\vec{w}}{|\vec{w}|} = \boxed{\pm (0, 1, 0)}$$

Note This method works for any $\vec{w}$ with $\vec{v}\cdot\vec{w} = 0$. For example, you may set $a = c = 2$, $b = 1$ and get $\vec{w} = (2, 1, 2)$, which yields $\pm \frac{\vec{w}}{|\vec{w}|} = \pm \frac{1}{3}(2, 1, 2)$

Ex Consider the points $P = (1,1,1)$, $Q = (3,1,2)$, $R = (1,4,0)$.

(1) Find the two unit vectors which are perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

Sol We take the unit vector of $\overrightarrow{PQ} \times \overrightarrow{PR}$

($\because$ $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$)

$$\overrightarrow{PQ} = (2,0,1), \quad \overrightarrow{PR} = (0,3,-1)$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (2,0,1) \times (0,3,-1) = (-3,2,6)$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-3)^2 + 2^2 + 6^2} = 7$$

$$\rightsquigarrow \pm \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \boxed{\pm \frac{1}{7}(-3,2,6)}$$

Note You can also solve this problem using dot products, but with much more work.

(2) Find the area of the triangle PQR.

Sol $\text{Area} = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \boxed{\frac{7}{2}}$

Addendum for HW 1

The volume of the parallelepiped given by $\vec{u}, \vec{v}, \vec{w}$ is $|\vec{u} \cdot (\vec{v} \times \vec{w})|$